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Current algebras and differential geometry

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Abstract: We show that symmetries and gauge symmetries of a large class of 2-dimensional σ-models are described by a new type of a current algebra. The currents are labeled by pairs of a vector field and a 1-form on the target space of the σ-model. We compute the current-current commutator and analyse the anomaly cancellation condition, which can be interpreted geometrically in terms of *Dirac structures*, previously studied in the mathematical literature. *Generalized complex structures* correspond to decompositions of the current algebra into pairs of anomaly free subalgebras. σ-models that we can treat with our method include both physical and topological examples, with and without Wess-Zumino type terms.

Keywords: Conformal Field Models in String Theory, Anomalies in Field and String Theories, Differential and Algebraic Geometry.

We dedicate this article to Ludwig Faddeev on the occasion of his 70th birthday.

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1. Introduction

Current algebras originate in particle physics [1]. The minimal coupling in gauge theories has the form $A^a_\mu J^\mu_a$, where A^a_μ is the gauge field and $J^\mu_a = \mathrm{Tr}\; e_a(\bar\psi\gamma^\mu\psi)$ is the fermionic current. Here ψ and $\bar\psi$ are fermionic fields, γ^μ are γ-matrices and e_a are the Lie algebra generators, $[e_a, e_b] = -f^c_{ab}e_c$. The understanding of properties of the currents J^μ_a is an essential piece in understanding the gauge coupling. In particular, by computing the density-density commutators $[J^0_a(x), J^0_b(y)]$ one usually finds an expression of the follwoing type,

$$[J^0_a(x), J^0_b(y)] = f^c_{ab} J^0_c(x)\delta(x-y) + \text{anomalous terms}\,, \tag{1.1}$$

where anomalous terms contain derivatives of $\delta(x-y)$. In gauge theory, anomalous terms indicate that the gauge symmetry cannot be preseved at the quantum level. If the anomalous terms are absent, the currents J^0_a form a set of first class constraints which can be imposed on the quantum system [2, 3].

Non-abelian one-dimensional current algebras or Kac-Moody algebras,

$$[J_a(x), J_b(y)] = f^c_{ab} J_c(x)\delta(x-y) + \kappa\delta_{ab}\delta'(x-y)\,. \tag{1.2}$$

can be viewed as coming from the theory of (1+1)-dimensional fermions interacting with a non-abelian gauge field. The algebra (1.2) plays a crucial role in Conformal Field Theory as the symmetry of the Wess-Zumino-Witten model [4]. It is also subject of a well developed mathematical theory [5].

In this paper we shall construct new current algebras of the type (1.2) with an index a replaced by a pair formed by a vector field and a 1-form on a manifold M which serves as a target of a $1+1$-dimensional σ-model. These current algebras naturally arise in the description of symmetries and gauge symmetries of both topological and physical σ-models. In fact, several examples of our current algebra, including (1.2), were known before, but we now present a unifying picture for many different types of σ-models. As an interesting twist in our calculation, we find a relation between the new current algebras and *Courant brackets* studied in the mathematical literature [6]. Some relations between Courant brackets and current algebras have been previously considered in [7, 8] in the context of vertex algebras.

In section 2 we recall the lagrangian and hamiltonian description of several classes of 2-dimensional σ-models. Our list includes the WZW model, the gauged WZW model, Poisson and WZ-Poisson σ-models. In section 3 we compute the current-current commutator and express it in terms of the Courant bracket. While the commutator in the full current algebra always has an anomalous part, for subalgebras one can study the anomaly concellation condition. We find that it gives rise to *Dirac structures*. Two transversal Dirac structures form a *generalized complex structure* and give rise to a polarization of our current algebra into two anomaly free subalgebras.

2. 2-dimesional σ-models

2.1 Examples

In this paper we consider several classes of 2-dimensional σ-models. Some of them are of importance in string theory applications, and others are topological field theories of interest in mathematical physics and the theory of quantization.

Let Σ be the two-dimesnional world-sheet or space-time (either euclidean or lorentzian), and M the target manifold of the σ-model. On local charts, Σ has coordinates $x^\alpha, \alpha = 1, 2$ and M has coordinates $X^i, i = 1, \ldots, \dim M$. A typical example of a σ-model is defined by metrics $h_{\alpha\beta}$ and G_{ij} on the world-sheet and on the target space, respectively,

$$S = \int_\Sigma \frac{1}{2} G_{ij}(X)\, \mathrm{d}X^i \wedge *\mathrm{d}X^j \,, \tag{2.1}$$

where $*\alpha$ is the Hodge dual of α with respect to h (and thus $\mathrm{d}X^i \wedge *\mathrm{d}X^j = \partial_\alpha X^i \partial^\alpha X^i \mathrm{d}\,\mathrm{vol}_\Sigma$, with $\partial^\alpha X^i \equiv h^{\alpha\beta}\partial_\beta X^i$ and $\mathrm{d}\,\mathrm{vol}_\Sigma \equiv \sqrt{|\det h|}\,\mathrm{d}^2 x$). Such σ-models arise in the theory of bosonic strings (or as bosonic parts of super-string actions) as well as in the theory of integrable models (e.g. the 2-dimensional $O(3)$ σ-model).

Given a 2-form B on the target space M one can complement the action (2.1) as follows,

$$S[X] = \int_\Sigma \frac{1}{2} G_{ij}(X)\, \mathrm{d}X^i \wedge *\mathrm{d}X^j + \int_\Sigma \frac{1}{2} B_{ij}(X)\, \mathrm{d}X^i \wedge \mathrm{d}X^j \,. \tag{2.2}$$

Here the second term is an integral over the world-sheet of the pull-back X^*B with respect to the map $X\colon \Sigma \to M$. More generally, given a closed 3-form H on M one can add a

Wess-Zumino term to the action (2.1),

$$S[X] = \int_\Sigma \frac{1}{2} G_{ij}(X)\,\mathrm{d}X^i \wedge *\mathrm{d}X^j + \int_N H\,, \tag{2.3}$$

where N is a 3-dimensional submanifold of M with $\partial N = X(\Sigma)$. In the string theory context, B is the NS-NS 2-form, and H is the corresponding field strength.

An interesting example of (2.3) is the Wess-Zumino-Witten (WZW) model [9], where $M = G$ is a Lie group with an invariant metric (for instance, G semi-simple with metric given by the Killing form) and H the Cartan 3-form. More explicitly, the action is given by

$$S[g] = \frac{k}{8\pi}\int_\Sigma \mathrm{Tr}\,(g^{-1}\mathrm{d}g \wedge *g^{-1}\mathrm{d}g) + \frac{k}{12\pi}\int_N \mathrm{Tr}(g^{-1}\mathrm{d}g)^3. \tag{2.4}$$

Here $g : \Sigma \to G$ and we denote the metric on the Lie algebra $\mathfrak{g}$ by Tr.

There is another class of σ-models which arises when instead of a metric G on M we are given a bi-vector $\Pi = \frac{1}{2}\Pi^{ij}(X)\partial_i \wedge \partial_j$. Then, the action is given by

$$S[A, X] = \int_\Sigma \left(A_i \wedge \mathrm{d}X^i + \frac{1}{2}\Pi^{ij}(X) A_i \wedge A_j \right), \tag{2.5}$$

where $A_i = A_{i\alpha}(x)\mathrm{d}x^\alpha$ and $A_{i\alpha}$ are components of 1-forms both on the world-sheet and on the target space. This model is topological if Π is a Poisson bi-vector, i.e. $\Pi^{ij} := \{X^i, X^j\}$ are Poisson brackets on M. Then, it is called a Poisson σ-model [10]–[12].

Similar to σ-models defined by a metric, one can add a Wess-Zumino term to the action (2.5),

$$S[A, X] = \int_\Sigma A_i \wedge \mathrm{d}X^i + \frac{1}{2}\Pi^{ij}(X) A_i \wedge A_j + \int_N H\,. \tag{2.6}$$

This action defines a topological field theory (the space of classical solutions modulo gauge symmetries is finite dimensional), iff (Π, H) defines a WZ-Poisson structure, i.e. if

$$\Pi^{il}\partial_l \Pi^{jk} + \mathrm{cycl}(ijk) = \Pi^{ii'}\Pi^{jj'}\Pi^{kk'} H_{i'j'k'} \tag{2.7}$$

holds true. (In terms of the bivector Π this may be rewritten as $\frac{1}{2}[\Pi, \Pi] = \langle H, \Pi \otimes \Pi \otimes \Pi \rangle$). Then the action (2.6) defines the WZ-Poisson σ-model [13] (cf. also [14]).

The WZW model can be turned into a topological theory [15] too, the so-called G/G model, by adding to the action (2.4) an extra piece reminiscent of (2.5),

$$\Delta S[g, a] = \frac{k}{4\pi}\int_\Sigma \mathrm{Tr}\left(a \wedge (* - 1)\mathrm{d}g g^{-1} - a \wedge (* + 1) g^{-1}\mathrm{d}g - a \wedge (* - 1) g a g^{-1} + a \wedge *a\right), \tag{2.8}$$

where a is a $\mathfrak{g}$-valued connection 1-form and, as before, $*$ denotes the Hodge duality operator with respect to the world-sheet metric h.

2.2 Hamiltonian formulation

All the models listed above share the following phase space description. For $\Sigma = S^1 \times \mathbb{R}$ the phase space is the cotangent bundle T^*LM of the loop space LM. Using local coordinates $X^i(\sigma)$ and their canonical conjugates $p_i(\sigma)$, the canonical symplectic form of the cotangent bundle is given by

$$\omega = \oint_{S^1} \delta X^i(\sigma) \wedge \delta p_i(\sigma)\, \mathrm{d}\sigma\,, \tag{2.9}$$

where δ denotes the de Rham differential on the phase space T^*LM.

Twisting (2.9) by a closed 3-form H on M, gives

$$\omega = \oint_{S^1} \delta X^i(\sigma) \wedge \delta p_i(\sigma)\, \mathrm{d}\sigma + \frac{1}{2}\oint_{S^1} H_{ijk}(X(\sigma))\, \partial X^i(\sigma)\, \delta X^j(\sigma) \wedge \delta X^k(\sigma)\, \mathrm{d}\sigma\,, \tag{2.10}$$

where ∂ is the derivative with respect to σ.

From (2.10) we read off the Poisson brackets

$$\{X^i(\sigma), X^j(\sigma')\} = 0\,, \qquad \{X^i(\sigma), p_j(\sigma')\} = \delta^i_j \delta(\sigma - \sigma') \tag{2.11}$$

$$\{p_i(\sigma), p_j(\sigma')\} = -H_{ijk}\partial X^k \delta(\sigma - \sigma')\,. \tag{2.12}$$

In the case of (2.1) this phase space is complemented by the specification of a hamiltonian

$$\mathcal{H} = \frac{1}{2}\oint_{S^1} \left(G^{ij}(X)p_i p_j + G_{ij}(X)\partial X^i \partial X^j\right)\, \mathrm{d}\sigma\,, \tag{2.13}$$

where G^{ij} denotes the inverse to G_{ij}. If the metric G admits Killing vector fields v_a, the action functional (2.1), or likewise the symplectic form (2.9) and the hamiltonian (2.13), have a symmetry generated by the Noether currents

$$J_a(\sigma) = (v_a)^i(X(\sigma))\, p_i(\sigma)\,. \tag{2.14}$$

In the case of the WZ-Poisson σ-model, eq. (2.6), the components of A_i along the ("spatial") circle S^1 become the momenta p_i, and the hamiltonian takes the form

$$\mathcal{H} = \oint_{S^1} \lambda_i \left(\partial X^i + \Pi^{ij}(X)p_j\right)\, \mathrm{d}\sigma\,. \tag{2.15}$$

Here $\lambda_i(\sigma)$ are (undetermined) Lagrange multipliers, the "time" componentes of A_i. Such a hamiltonian enforces that the currents

$$J^i = \partial X^i + \Pi^{ij} p_j \tag{2.16}$$

on T^*LM vanish.They are the constraints of the hamiltonian system corresponding to (2.6).

The functionals (2.14) and (2.16) are two particular examples of the following type. Choose a vector field $v = v^i(X)\partial_i$ and a 1-form $\alpha = \alpha_i(X)\mathrm{d}X^i$ on M, and associate to them a current,

$$J_{(v,\alpha)}(\sigma) = v^i(X(\sigma))p_i(\sigma) + \alpha_i(X(\sigma))\partial X^i\,. \tag{2.17}$$

Likewise, consider a WZ-type σ-model as in (2.3), leading to the symplectic form (2.10). Assuming that v is a Killing vector field for the metric G_{ij} which preserves the 3-form H, the Noether current need not exist. There is an extra condition which requires that the contraction of v with H is not only closed but exact, i.e. that there exists some 1-form $\alpha = \alpha_i \mathrm{d}X^i$ on M such that

$$v^i H_{ijk} = \partial_j \alpha_k - \partial_k \alpha_j\,. \tag{2.18}$$

If this condition is satisfied, the Noether current is precisely $J_{(v,\alpha)}$.[1] There is an ambiguity in choosing 1-forms α solving equation (2.18), but this is all the ambiguity for the Noether current corresponding to v. This situation generalizes in a straightforward way to the presence of several Killing vector fields leaving (2.3) invariant. In particular, in the WZW model (2.4) the left and right chiral currents are of the form,

$$J^L = p - \frac{k}{4\pi} g^{-1}\partial g\,, \qquad J^R = gpg^{-1} + \frac{k}{4\pi}\partial g g^{-1}\,. \tag{2.19}$$

Here $p = p(\sigma)$ is a left-invariant momentum (a Lie algebra valued matrix of momenta).

Investigating the Poisson brackets and commutation relations of $J_{(v,\alpha)}$ will be one of the main goals of this paper.

3. Current algebra

3.1 Current algebra and Courant bracket

In this section we present the computation of the Poisson bracket $\{J_{(u,\alpha)}, J_{(v,\beta)}\}$ between currents, with u, v two vector fields and α, β two 1-forms. Presenting the answer requires the following two structures.

First, we need a symmetric scalar product on the space of vector fields and 1-forms,

$$\langle (u,\alpha),(v,\beta)\rangle_+ = \alpha(v) + \beta(u)\,. \tag{3.1}$$

This scalar product associates to two pairs (u,α) and (v,β) a function of X^i. Note that the right hand side of (3.1) can be both positive or negative.

The second structure is known as a *Courant bracket* [16]–[18] and it associates to the two pairs (u,α) and (v,β) another pair of the same type,

$$[(u,\alpha),(v,\beta)] = ([u,v]_{\rm Lie}, L_u\beta - L_v\alpha + {\rm d}\,(\alpha(v)) + H(u,v,\cdot))\,. \tag{3.2}$$

Here $[u,v]_{\rm Lie}$ is the Lie bracket of the vector fields u and v, L_u, L_v stand for Lie derivatives with respect to u and v, respectively, and $H(u,v,\cdot)$ is a 1-form obtained by contracting H with u and v. The bracket (3.2) is not skew-symmetric. It has many interesting properties, the most interesting one being the Leibniz identity,

$$[(u,\alpha),[(v,\beta),(w,\gamma)]] = [[(u,\alpha),(v,\beta)],(w,\gamma)] + [(v,\beta),[(u,\alpha),(w,\gamma)]]\,. \tag{3.3}$$

This equation is a counterpart of the Jacobi identity for non skew-symmetric brackets.

We are now ready to present the formula for a Poisson bracket of two currents,

$$\{J_{(u,\alpha)}(\sigma), J_{(v,\beta)}(\tau)\} = -J_{[(u,\alpha),(v,\beta)]}(\sigma)\delta(\sigma-\tau) + \langle (u,\alpha),(v,\beta)\rangle_+(X(\tau))\,\delta'(\sigma-\tau)\,. \tag{3.4}$$

This expression shows that the currents $J_{(u,\alpha)}$ form a current algebra, with the anomalous contribution governed by the scalar product, and with the linear in J contribution given by the Courant bracket.

[1] Associating a pair (v,α) (rather than only a vector field v) to a symmetry of a 2-dimensional σ-model is one of the messages of Letter 1 in [16].

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For completeness we also compute Poisson brackets between currents and functions on the target space,

$$\{f(X(\tau)), J_{(u,\alpha)}(\sigma)\} = u(f)(X(\tau))\delta(s-\tau)\,. \tag{3.5}$$

This equation together with the Leibniz identity for the Courant bracket ensures the Jacobi identity of the bracket (3.4). Note that the currents $J_{(u,\alpha)}(\sigma)$ and $f(X(\tau))$ are not independent: $\partial f = J_{(0,\mathrm{d}f)}$. Using test functions $\epsilon(\sigma)$, this linear dependence may be expressed also as

$$\oint \left[\epsilon(\sigma)J_{(0,\mathrm{d}f)}(\sigma) + (\partial\epsilon)(\sigma)f(X(\sigma))\,\mathrm{d}\sigma\right] \equiv 0\,. \tag{3.6}$$

It should be mentioned that the bracket (3.4) can be presented in many different ways by changing the argument in the anomalous term, e.g.

$$\begin{aligned}\{J_{(u,\alpha)}(\sigma), J_{(v,\beta)}(\tau)\} \;=\; &-J_{[(u,\alpha),(v,\beta)]-\mathrm{d}\langle(u,\alpha),(v,\beta)\rangle_+}(\sigma)\delta(\sigma-\tau)\,+\\ &+\langle(u,\alpha),(v,\beta)\rangle_+(X(\sigma))\delta'(\sigma-\tau)\end{aligned} \tag{3.7}$$

or

$$\begin{aligned}\{J_{(u,\alpha)}(\sigma), J_{(v,\beta)}(\tau)\} \;=\; &-J_{[(u,\alpha),(v,\beta)]-\frac{1}{2}\mathrm{d}\langle(u,\alpha),(v,\beta)\rangle_+}(\sigma)\delta(\sigma-\tau)\,+\\ &+\langle(u,\alpha),(v,\beta)\rangle_+\left(X\left(\frac{1}{2}(\sigma+\tau)\right)\right)\delta'(\sigma-\tau)\,.\end{aligned} \tag{3.8}$$

Note that the bracket that now appears in the argument of the currents,

$$[(u,\alpha),(v,\beta)]_{\mathrm{skew}} = [(u,\alpha),(v,\beta)] - \frac{1}{2}\mathrm{d}\langle(u,\alpha),(v,\beta)\rangle_+$$

is skew-symmetric. This is the consequence of antisymmetry of the Poisson brackets (2.11) and (2.12) underlying the current algebra. With this bracket, however, the nice property (3.3) is replaced by a homotopy Jacobi identity with the right hand side given by an exact form (for details see [6]).

One can treat the linear relation (3.6) in a slightly different way by adding the two types of currents, $J_{(v,\alpha)}(\sigma)$ and $f(X(\sigma)))$, corresponding to an abelian extension of the J-current algebra. The extended currents now form a Lie algebra, obtained from (3.4) and (3.5) above. Smearing them by means of test functions, and using $C^\infty(M)\otimes C^\infty(S^1) \cong C^\infty(M\times S^1)$, one is then lead to consider

$$J_\psi = \oint \left(v^i(X(\sigma),\sigma)p_i(\sigma) + \alpha_i(X(\sigma),\sigma)\partial X^i + f(X(\sigma),\sigma)\right)\mathrm{d}\sigma \tag{3.9}$$

as extended currents. Here ψ may be interpreted as a section of $\widetilde{E} := T\widetilde{M}\oplus T^*\widetilde{M}$, $\widetilde{M}\equiv M\times S^1$, whose tanget vector part is parallel to M: $\psi = v^i(X,\sigma)\partial_i + \alpha_i(X,\sigma)\mathrm{d}X^i + f(X,\sigma)\mathrm{d}\sigma$. The kernel of the map $\psi \mapsto J_\psi \in C^\infty(T^*LM)$ is provided by the exact 1-forms on $\widetilde{M}$, $J_{\widetilde{\mathrm{d}}f}\equiv 0$, where $\widetilde{\mathrm{d}} = \mathrm{d}+\mathrm{d}\sigma\wedge\partial$ is the de Rham differential on $\widetilde{M}$; this just reexpresses (3.6). The Lie algebra of the extended currents (3.9) may now be cast into the following simple form:

$$\{J_{\psi_1}, J_{\psi_2}\} = -J_{[\psi_1,\psi_2]}\,,$$

where $[\psi_1, \psi_2]$ denotes the Courant bracket in $\widetilde{E}$. Indeed, modulo exact terms the Courant bracket becomes antisymmetric, cf. eq. (3.2), so that the quotient algebra is a Lie algebra on behalf of (3.3).[2] Thus the map $J.$ is an (anti-)isomorphism from the Lie algebra constructed from the Courant bracket on $\widetilde{E}$ as described above to our "current" Lie algebra, realized as Poisson subalgebra in loop phase space T^*LM.[3]

The current-current brackets (3.4) (or (3.7), (3.8)) resemble anomalous commutators in (3+1) dimensions [19, 2, 3]:

$$[J_a^0(x), J_b^0(y)] = f_{ab}^c J_c^0(x)\delta(x-y) + d_{abc}\epsilon_{ijk}\partial_i A_j^c(x)\partial_k \delta(x-y)\,.$$

Here A_j^a is the background Yang-Mills field, and d_{abc} are symmetric structure constants $d_{abc} = 1/2\ \mathrm{Tr}\,(e_a e_b + e_b e_a)e_c$. Similar to (3.4), the coefficient in front of the derivartive of the δ-function is a field with a nontrivial x-dependence.

The only piece of data that we used to define the current algebra was a closed 3-form H on M. In fact, the current algebra only depends on the cohomology class of H. Indeed, for two choices of a 3-form, H and $H' = H + dB$, the Poisson brackets (2.11), (2.12) are related to one another by a simple (non-canonical) transformation

$$X^i(\sigma) \mapsto X^i(\sigma)\,,\; p_i(\sigma) \mapsto p_i(\sigma) + B_{ij}(X(\sigma))\partial X^j\,. \tag{3.10}$$

The corresponding transformation on currents reads

$$J_{(v,\alpha)} \mapsto J_{(v,\alpha+B(v,\cdot))}\,.$$

This effect has a counterpart in mechanics of a charged particle in a magnetic field. The transformation (3.10) is analogous to the passage from canonical momenta p to kinetic momenta $\pi = p - \frac{e}{c}A$ for an ordinary particle in a magnetic field $B = \mathrm{d}A$. Implementing this change of variables in the canonical symplectic form $\omega = \mathrm{d}q^i \wedge \mathrm{d}p_i$, one obtains

$$\omega = \mathrm{d}q^i \wedge \mathrm{d}\pi_i - \frac{e}{c}B\,, \tag{3.11}$$

which resembles the symplectic form (2.10) on loop space we started with. Similarly, by a shift of variables as in (3.10) we can eliminate any exact H in (2.10) altogether (cf. also [13]).

Finally we remark that since all J's are at most linear in the momenta p_i one can consistently replace Poisson brackets by commutators in all formulas above.

3.2 Dirac structures and examples in physics

The current algebra (3.4) is very big since it allows for a choice of arbitrary vector fields and 1-forms. So, it makes sense to look for some interesting subalgebras which are somewhat smaller. In particular, one can ask when J's form a Lie algebra with no anomaly term. This requires two conditions: first, all pairs (u, α) in such a subalgebra should have vanishing scalar products, $\langle (u_1, \alpha_1), (u_2, \beta_2)\rangle = 0$. Second, the Courant brackets should close on the space of such pairs. If in addition (u, α)'s span a dimension $n = \dim M$ subbundle of $TM \oplus T^*M$, this is called a Dirac structure on M [17, 6].

[2] This quotient Lie algebra is certainly not C^∞-linear, so it cannot arise from a Lie algebroid.

[3] We are grateful to the Referee for suggesting this perspective.

As the first example, let us return to the WZ-type model (2.3) with a Killing vector field v satisfying equation (2.18) for some 1-form α. Then, the Noether current $J_{(v,\alpha)}(\sigma)$ has an anomalous Poisson bracket,

$$\{J_{(v,\alpha)}(\sigma), J_{(v,\alpha)}(\tau)\} = J_{(0,\mathrm{d}(v^i\alpha_i))}(\sigma)\delta(\sigma-\tau) + (v^i\alpha_i)(X(\tau))\delta'(\sigma-\tau)\,.$$

Vanishing of the anomaly gives a new condition

$$\alpha(v) \equiv v^i\alpha_i = 0\,. \tag{3.12}$$

Together with condition (2.18) this is tantamount to the 3-form H extending to an equivariant 3-form, $(\mathrm{d}-\iota_v)(H+\alpha)=0$, where ι_v is the contraction with respect to the vector field v (for a definition of equivariant forms see [20]). In a similar fashion, if there are several Killing vector fields, forming a Lie algebra $[v_a, v_b] = -f^c_{ab}v_c$, the absence of the δ'–contribution requires $\iota(v_a)\alpha_b + \iota(v_b)\alpha_a = 0$ for all a and b. In this case, closure of the bracket is not automatic: in addition one needs a choice of α's such that $L_{v_a}\alpha_b = -f^c_{ab}\alpha_c$. Then, the currents $J_a = J_{(v_a,\alpha_a)}$ form a Lie algebra

$$\{J_a(\sigma), J_b(\tau)\} = f^c_{ab}J_c(\sigma)\delta(\sigma-\tau)\,. \tag{3.13}$$

Again, these conditions may be summarized compactly as saying that H should extend to an equivariantly closed 3-form.

Note that even if there is an anomaly in the current-current Poisson bracket (or commutator), the Noether charges $Q_a = \oint J_a(\sigma)\,\mathrm{d}\sigma$ always form a representation of the Lie algebra; the anomalous term-such as any exact piece in the 1-form part of the Courant bracket-cancels by integrating over the circle.

Anomaly free Noether currents are needed if one wants to gauge a given rigid symmetry. The anomaly cancellation condition is equivalent to requiring that the currents are first class constraints. Obstructions in gauging WZ-type σ-models were analyzed from a lagrangian perspective in [21] and related to equivariant cohomology in [22]. In our approach, the anomaly cancellation conditions read $H(v_a,\cdot,\cdot) = \mathrm{d}\alpha_a$ with $\alpha_a(v_b) + \alpha_b(v_a) = 0$.

In the WZW model, the currents (2.19) form the standard one-dimensional current algebra,

$$\begin{aligned}
\{J^L_a(\sigma), J^L_b(\tau)\} &= f^c_{ab}J^L_c(\sigma)\delta(\sigma-\tau) + \frac{k}{2\pi}\delta_{ab}\delta'(\sigma-\tau)\,,\\
\{J^R_a(\sigma), J^R_b(\tau)\} &= f^c_{ab}J^R_c(\sigma)\delta(\sigma-\tau) - \frac{k}{2\pi}\delta_{ab}\delta'(\sigma-\tau)\,,\\
\{J^L_a(\sigma), J^R_b(\tau)\} &= 0\,.
\end{aligned} \tag{3.14}$$

By Fourier decomposition one obtains the more familiar form of the Kac-Moody algebra. The combination $J_a = J^R_a - J^L_a$ is anomaly free and can be gauged out. In fact, this is exactly the constraint of the gauged WZW model (2.4), (2.8),

$$J = gpg^{-1} - p + \frac{k}{4\pi}\left(\partial g g^{-1} + g^{-1}\partial g\right).$$

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The corresponding Dirac structure is formed by pairs

$$v = x^R - x^L\,, \qquad \alpha = \frac{k}{4\pi}\,\mathrm{Tr}\; x(dgg^{-1} + g^{-1}dg)\,,$$

where x is an element of the Lie algebra and x^L and x^R are the corresponding left- and right-invariant vector fields on the group G.

The other example we want to discuss are the constraints (2.16) of Poisson σ-models and WZ-Poisson σ-models. These models are topological if all of the constraints are of the first class. Consider the actions (2.5) and (2.6) with no restriction on the background field Π^{ij}. At the hamiltonian level one can even relax the condition that Π^{ij} be skew-symmetric. Then, according to the above considerations, the constraints (2.16) are of the first class iff the pairs $(v = \Pi^{ij}\alpha_i\partial_j, \alpha = \alpha_i dX^i)$ form a Dirac structure. In the case of the action (2.5) this amounts to the tensor Π^{ij} being skew-symmetric and Poisson. The first condition comes from isotropy of D with respect to the scalar product (3.1) and ensures vanishing of the anomaly in the current algebra. The second condition results from the closedness of sections of D with respect to the Courant bracket (3.2) (with $H = 0$) and corresponds to the closedness of current-current Poisson brackets or commutators.

Now let us turn to the more general action (2.6). Again we can start with the currents (2.16), possibly dropping the requirement that Π^{ij} is antisymmetric, and pose the question under what conditions they can be used as first class constraints. We see that this is equivalent to requiring that the graph of the two-tensor Π defines a Dirac structure. The isotropy requirement is the same as before (since the inner product (3.1) is unchanged) and thus is satisfied iff Π^{ij} is skew-symmetric. But the closure condition of the Courant bracket now shows a nontrivial effect from the contribution of H; as a result one finds that the Jacobiator of Π^{ij} does not vanish anymore, but fulfills equation (2.7) [13, 23].

As a possible physical realization of WZ-Poisson structures we return to a point particle of mass m in a magnetic field B.[4] In the presence of a magnetic charge density ρ_m, eq. (3.11) defines a non-degenerate 2-form ω which is not closed, $\mathrm{d}\omega = -(e/c)\,\mathrm{d}B \propto \rho_m\,\mathrm{d}^3q \neq 0$. Let Π be the (negative) inverse of ω. It is no longer Poisson, but is easily seen to satisfy (2.7) with $H \propto \mathrm{d}B$. Letting as usual the hamiltonian $\mathcal{H} = \vec{\pi}^{\,2}/2m$, one obtains the vector field $V_{\mathcal{H}} = -\Pi(\mathrm{d}\mathcal{H}, \cdot)$. The corresponding dynamical system $(\dot{q}, \dot{\pi}) = V_{\mathcal{H}}$ reproduces the equations of motion of a point particle under the influence of the Lorentz force generated by B.

3.3 Generalized complex structures

The labels of the current algebra (α, v) can be extended to complex valued 1-forms and vector fields on M. Both the scalar product (3.1) and the Courant bracket (3.2) extend in a natural way. The form of equation (3.4) remains the same as well. This simple extension leads to the notion of *generalized complex structures* recently introduced in [25] (cf. also [26]).

Let $E^{\mathbb{C}} = T^{\mathbb{C}}M \oplus (T^*)^{\mathbb{C}}M$ be the complexified Courant algebroid. A generalized complex structure is a smooth family of operators $J_X : E^{\mathbb{C}}_X \to E^{\mathbb{C}}_X, X \in M$, with the

[4]T.S. thanks R. Jackiw for drawing his attention to [24], which motivated the consideration below.

following properties. First, $J^2 = -1$, so that $E^{\mathbb{C}}$ splits into two subbundles $E^{\mathbb{C}} = E^{\mathbb{C}}_+ \oplus E^{\mathbb{C}}_-$, corresponding to the eigenvalues i and $-i$ of J. Second, the subbundles $E^{\mathbb{C}}_+$ and $E^{\mathbb{C}}_-$ are both Dirac structures in $E^{\mathbb{C}}$. In other words, a generalized complex structure is a splitting of $E^{\mathbb{C}}$ into a sum of two complementary Dirac subbundles.

In particular, ordinary complex structures correspond to splittings of the form $E^{\mathbb{C}}_+ = T^{(1,0)}M \oplus (T^{(0,1)})^*M$ and $E^{\mathbb{C}}_- = T^{(0,1)}M \oplus (T^{(1,0)})^*M$. Another type of examples is given by symplectic structures. In this case, $E^{\mathbb{C}}_+$ consists of pairs $(v, \omega(v,\cdot))$ and $E^{\mathbb{C}}_-$ of pairs $(v, -\omega(v,\cdot))$.

In terms of the current algebra, a generalized complex structure gives a splitting of all currents into two anomaly free subalgebras, such that the anomaly terms arise only in the Poisson brackets of currents from two different subalgebras.

3.4 D-branes

In the case of open strings or worldsheets with boundaries, additional input is necessary. We do not discuss this question on the level of action functionals such as (2.3), but instead turn directly to the hamiltonian picture. Our phase space now is the cotangent bundle T^*PM of paths in M, with endpoints attached to D-branes $D_0, D_1 \subset M$. So, $X^i(\sigma)$ is a map from $[0,1]$ to M such that $X^i(0) \in D_0$ and $X^i(1) \in D_1$. The canonical symplectic 2-form is

$$\omega = \int_0^1 \delta X^i(\sigma) \wedge \delta p_i(\sigma)\, \mathrm{d}\sigma\,. \tag{3.15}$$

In order to define an analogue of the twisted symplectic 2-form (2.10) for T^*PM, in addition to the closed 3-form H we need primitives B^0 and B^1 on the D-branes, i.e. a 2-form B^0 on D_0 satisfying $\mathrm{d}B^0 = H|_{D_0}$, and likewise so for D_1. Then

$$\begin{aligned}\omega = &\int_0^1 \delta X^i(\sigma) \wedge \delta p_i(\sigma)\, \mathrm{d}\sigma + \frac{1}{2}\int_0^1 H_{ijk}(X(\sigma))\, \partial X^i(\sigma)\, \delta X^j(\sigma) \wedge \delta X^k(\sigma)\, \mathrm{d}\sigma + \\ &+ \frac{1}{2} B^0_{ij}(X(0))\, \delta X^i(0) \wedge \delta X^j(0) - \frac{1}{2} B^1_{ij}(X(1))\, \delta X^i(1) \wedge \delta X^j(1)\end{aligned} \tag{3.16}$$

defines a symplectic 2-form on T^*PM. Note that the boundary contributions are needed in verifying the closedness condition for ω.

Given a current $J_{(v,\alpha)}$ we need to decide whether the boundary conditions imposed by D-branes D_0, D_1 preserve the symmetry generated by this current. From a mathematical point of view, this amounts to checking whether the differential $\delta J_{(v,\alpha)}$ can be obtained by inserting some vector in the 2-form ω. This gives two conditions. First, the vector field v should be tangent to the D-branes D_0 and D_1. Second, the 1-form $\alpha + B^0(v,\cdot)$ should vanish on D_0 while the 1-form $\alpha + B^1(v,\cdot)$ should vanish on D_1.

As a first example we consider the Poisson σ-model (2.5). Here, $H \equiv 0 \equiv B^0 \equiv B^1$. The constraints are again of the form (2.17) with the condition that everywhere $(\alpha, v) = (\alpha, \Pi(\alpha,\cdot))$ and that on the boundary v is tangent to the respective D-brane for any α that vanishes upon restriction to it, i.e. for any α in the conormal bundle to the brane. Describing the respective D-brane (locally) as the level zero set of some functions f^I, where $I = 1, \ldots, \dim M - \dim D$, the set of these α's is spanned by $\mathrm{d}f^I$. The condition to be satisfied is then that $\Pi(\mathrm{d}f^I, \cdot)) \equiv \{f^I, \cdot\}$ needs to be parallel to the surfaces $f^I = 0$. This is recognized as the first class property of such surfaces. So, in agreement with [27] we find that admissible D-branes of maximal symmetry in the Poisson σ-model should be first class or coisotropic submanifolds of the Poisson manifold M. But also other D-branes are conceivable, cf. [28], restricting permitted α's to a subset of elements of the conormal bundle of the brane (such that $v = \Pi(\alpha, \cdot)$ is still in its tangent bundle); they are thus recognized as branes of less symmetry.

As a slightly more complicated example, we consider the WZW model with D-branes D_0 and D_1 two conjugacy classes in G. Then, the symmetries generated by left- and right-moving currents J_a^L and J_a^R are broken since the left- and right-invariant vector fields are not tangent to conjugacy classes. But the combination $J_a = J_a^L - J_a^R$ corresponds to $v = e_a^L - e_a^R$ which is tangent to D_0 and D_1. If we choose B such that

$$B(x^L - x^R, \cdot) = \frac{k}{4\pi} \operatorname{Tr} x(dgg^{-1} + g^{-1}dg)\,,$$

the second condition will be fulfilled as well, and the symmetry generated by J_a's will be preserved by the D-branes.

4. Outlook

In this paper we gave a natural derivation of the Courant bracket in terms of a new type of current algebras. Moreover, Dirac structures correspond to anomaly free subalgebras of this current algebra, and generalized complex structures give rise to a splitting of our current algebra into pairs of anomaly free subalgebras. In fact, all axioms of the Courant bracket (or, better, the underlying Courant algebroid, provided one permits also degenerate inner products) can be shown to be equivalent to the properties satisfied by a current algebra of the kind introduced in this paper.

More complicated geometric structures can be induced by studying the current algebra including higher order derivatives. For example, the Poisson brackets will close for currents of the form,

$$J_\psi(\sigma) = v^i p_i(\sigma) + \alpha_i \partial X^i(\sigma) + \beta_i \partial^2 X^i(\sigma) + \gamma_{ij} \partial X^i \partial X^j\,,$$

where $\psi = (v, \alpha, \beta, \gamma)$. The Poisson brackets take the form,

$$\{J_\psi(\sigma), J_\phi(\tau)\} = -J_{[\psi,\phi]}(\sigma)\delta(\sigma-\tau) + J_{(\psi,\phi)}(\sigma)\delta'(\sigma-\tau) + J_{\langle\psi,\phi\rangle}(\sigma)\delta''(\sigma-\tau))\,.$$

Here one gets three different brackets, $[\psi, \phi]$ is an extension of the Courant bracket, (ψ, ϕ) is an extension of the Courant scalar product, and $\langle\psi, \phi\rangle$ is a new skew-symmetric scalar product. The geometric meaning of this (and higher) structures is not yet explored.

Recently, the Courant bracket attracted a lot of attention in connection with generalized complex geometry and supersymmetric σ-models (cf. e.g. [29]). It is natural to expect that our current algebra admits supersymmetric extensions which can be useful in this context. In the case of ordinary current algebras such extensions have been studied in [30].

In section 2 we provided a list of σ-models where examples for the currents (2.17) arise as constraints or symmetry generators. One may address the quest for further covariant two-dimensional models where such currents arise in this way. In [31] such a model is

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provided for any maximally istotropic subbundle D of $E = T^*M \oplus TM$. If D is a Dirac structure, one obtains a topological model generalizing the Poisson σ-model and the G/G WZW model, a Dirac σ-model.

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